Preferences under uncertainty

Theory of Individual and Strategic Decisions

MSc Human Decision Science

Maastricht University

- One of the first questions:

  How can we test the hypothesis that a person is risk-averse?
  Or more in general, how can we study risk attitudes?

- We will need two things:
  1. Find a choice domain X that involves uncertainty

     (Preview: this will be the set of lotteries)
  2. Have a model $U : X \to \mathbb{R}$ that allows us to formalize risk.

     (Preview: this will be Expected Utility)

- **Outcomes.** A finite set of sure monetary amounts,

$$Z = \{z_1, \ldots, z_n\}.$$

- For example, the payments a subject could in principle receive in an experiment.
- In the next few slides, I will take $Z = \{0, 4, 10\}$

- **Lotteries.** A lottery is a device/mechanism that randomly draws an outcome from Z.
- The probability $p(z)$ of each outcome z is:
  1. exogenously specified
  2. known by the agent (objective uncertainty).
- A lottery is written as follows:

$$p = \Big(p(0) \times 0,\ \ p(5) \times 4,\ \ p(10) \times 10\Big)$$

  In the textbook they write

$$p = \Big(p(0) \cdot 0 \oplus p(5) \cdot 4 \oplus p(10) \cdot 10\Big)$$

- Only the probabilities matter, not the experiment
- Every outcome z is a degenerate lottery

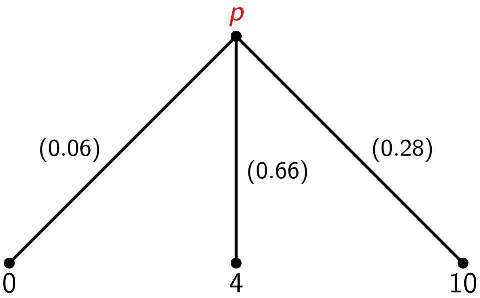
$$[z] = (1 \times z)$$

- The set of all lotteries over Z will be the set of alternatives:

$$X = L(Z)$$

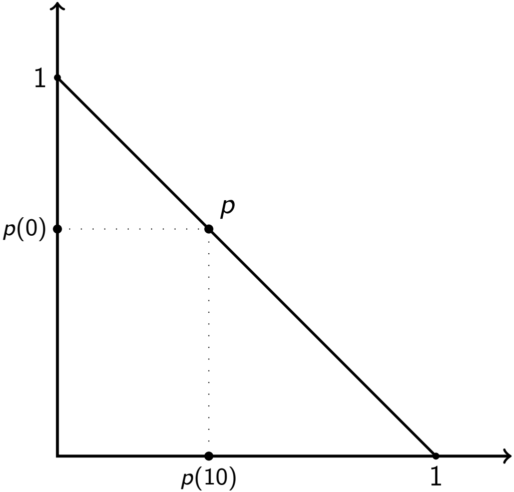
Example

The **lottery** $p = \left(0.06 \times 0,\ 0.66 \times 4,\ 0.28 \times 10\right)$ depicted:



Example

The set $L(Z)$ of all lotteries over $Z = \{0, 10\}$.



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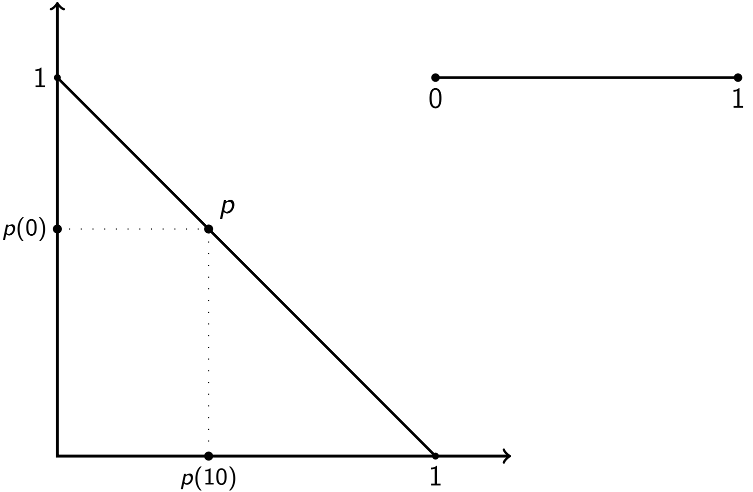

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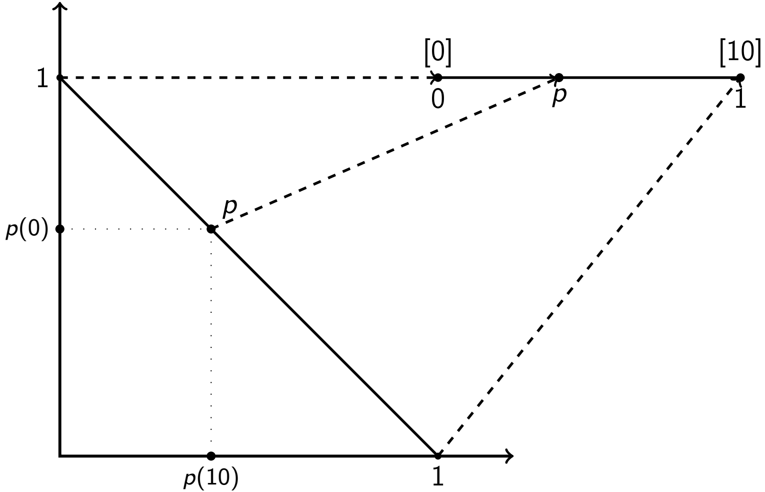

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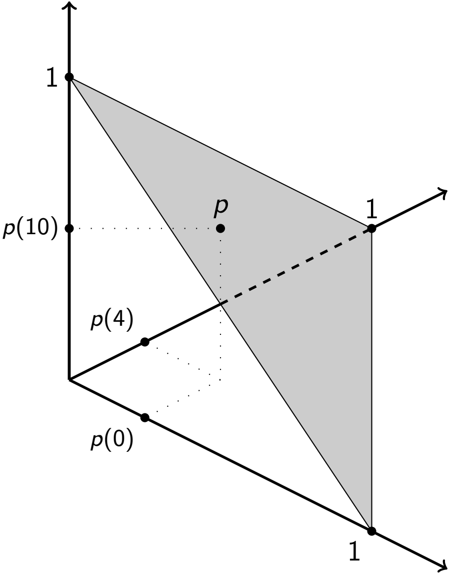

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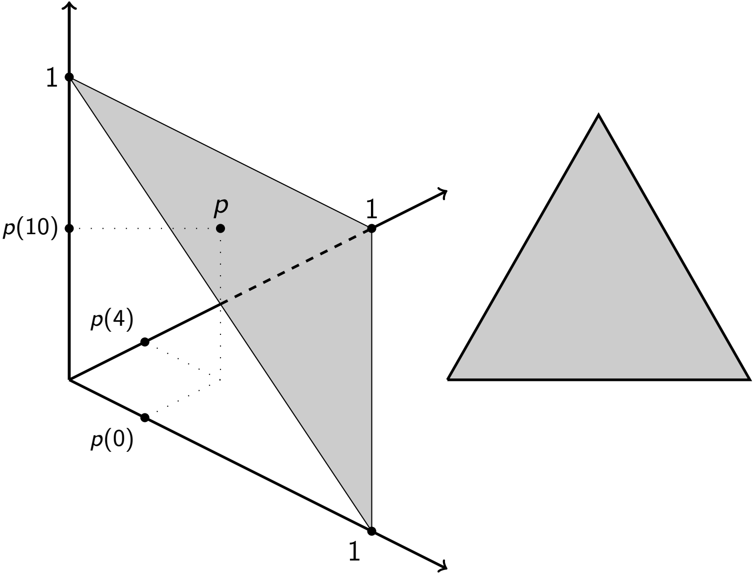

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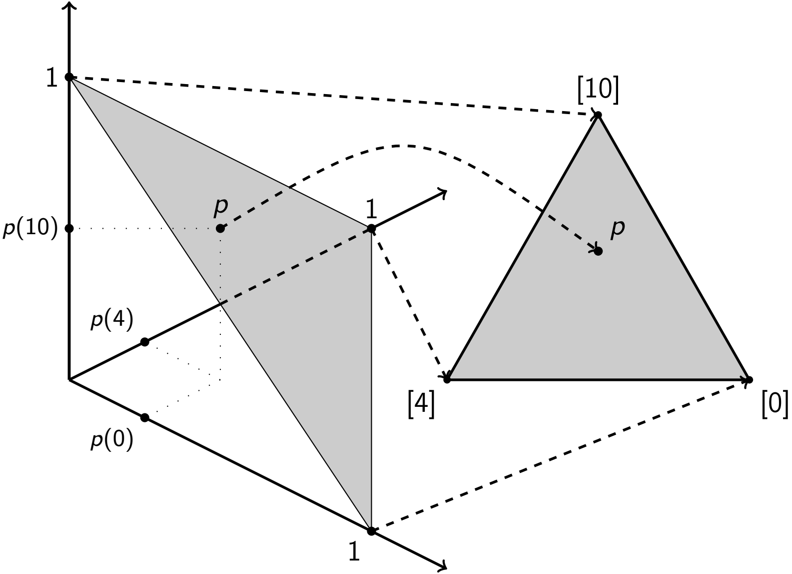
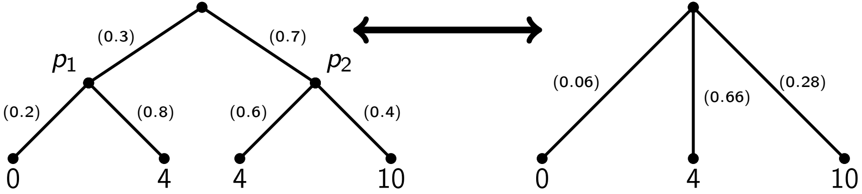
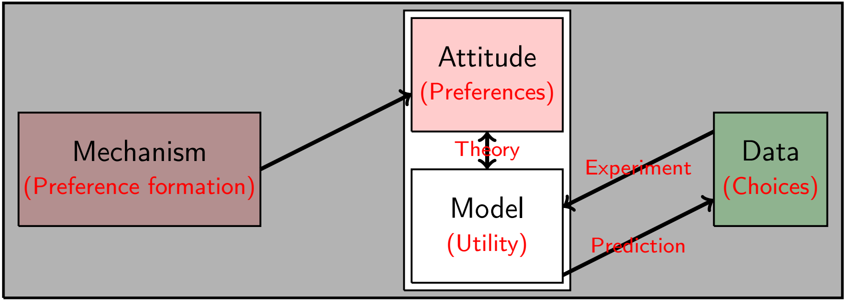


- A **compound lottery** $(0.3 \times p_1, 0.7 \times p_2)$ is a lottery over lotteries p_1 and p_2.
- The same as the simple lottery $(0.06 \times 0, 0.66 \times 4, 0.28 \times 10)$
- Remember, only probabilities of eventual outcomes matter



- We will build a model that represents preferences over lotteries
- This model will be called **Expected Utility**.



Mechanism

(Preference formation)

Attitude

(Preferences)

Theory

Model

(Utility)

Experiment

Prediction

Data

(Choices)

How would such a model look?

- Most intuitive way: evaluate lotteries by some kind of average
- But people do not necessarily maximize expected payment:

  What do you prefer, $p = (0.5 \times 10, 0.5 \times 0)$ or $q = [4]$?
- So, different form of average is needed

- Expected utility of lottery p is given by

$$U(p) = p(z_1)v(z_1) + \cdots + p(z_n)v(z_n)$$

- Two step procedure to evaluate lotteries:
  1. Assign a Bernoulli utility to each outcome
  2. Take weighted average of the Bernoulli utilities

Example

If $v(0) = 0$, $v(4) = 6$, $v(10) = 10$, our model evaluates the two earlier lotteries $p = (0.5 \times 10, 0.5 \times 0)$ and $q = [4]$ as follows:

$$\begin{aligned} U(p) &= 0.5 \cdot v(0) + 0.5 \cdot v(10) = 5 \\ U(q) &= 1 \cdot v(4) = 6. \end{aligned}$$

Hence, the model predicts that q is preferred. Does this say something about risk attitudes?

- Two important questions:
  1. What kind of preferences does U represent?
  2. Can we identify v?

- The individual has preferences $\succeq$ over $L(Z)$.
- The preferences are assumed to satisfy the following axioms:

1. **Completeness**: For any lotteries $p, q \in L(Z)$,
$$p \succeq q \text{ or } q \succeq p$$
2. **Transitivity**: For any lotteries $p, q, r \in L(Z)$,
$$\text{if } p \succeq q \text{ and } q \succeq r, \text{ then } p \succeq r.$$
3. **Continuity**: For any lotteries $p, q, r \in L(Z)$ with $p \succ q \succ r$,
$$\text{there is some } \alpha \in (0,1) \text{ such that } q \sim \big(\alpha \times p, (1-\alpha) \times r\big).$$
4. **Independence of irrelevant alternatives (IIA)**: For any lotteries $p, q, r \in L(Z)$ and any $\alpha \in [0,1]$,
$p \succeq q$ if and only if $\big(\alpha \times p, (1-\alpha) \times r\big) \succeq \big(\alpha \times q, (1-\alpha) \times r\big)$.

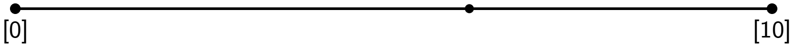
- These are called von Neumann-Morgenstern (vNM) axioms

- Take $Z = \{0, 4, 10\}$ and assume that $[10] \succ [4] \succ [0]$.
- The line below is the set of lotteries over $\{0, 10\}$:

Each $p = \big(\alpha \times 10, (1-\alpha) \times 0\big)$ is a point on this line

- Large α's are “good lotteries": better than [4]
- Small α's are “bad lotteries": worse than [4]
- Continuity says that the point where the green and the red segment meet is neither red nor green, but blue (indifference):

Indifference between lottery $p_0 = \big(0.6 \times 10, 0.4 \times 0\big)$ and [4]



The point of indifference is entirely subjective

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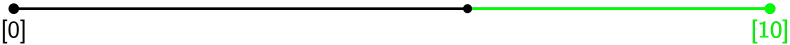

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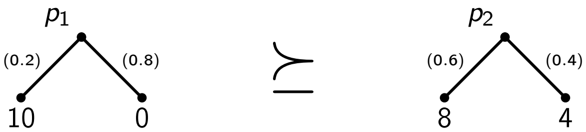
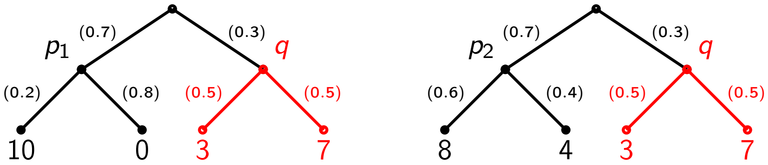
The point of indifference is entirely subjective

- Let $p_1 = (0.2 \times 10, 0.8 \times 0)$ and $p_2 = (0.6 \times 8, 0.4 \times 4)$.



- Let $p_1 = (0.2 \times 10, 0.8 \times 0)$ and $p_2 = (0.6 \times 8, 0.4 \times 4)$.
- Mix both with another lottery $q = (0.5 \times 3, 0.5 \times 7)$.



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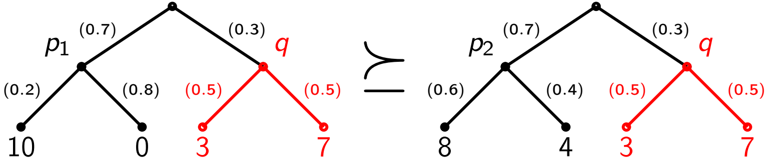
- IIA says that the preference relationship of the compound lotteries is not reversed



1. Background
2. Lotteries
3. Expected Utility: Preliminaries
4. Preferences over lotteries
5. **Expected Utility representation**
6. A well-known experiment
7. Risk attitudes
8. Homework

- Most important theorem in Decision Theory historically

Theorem (von Neumann & Morgenstern, 1944 (Part I))

The preferences $\succeq$ satisfy the four vNM axioms

if and only if

there is some Bernoulli utility function $v : Z \to \mathbb{R}$ such that for any two lottery $p, q \in L(Z)$ the following holds

$$p \succeq q \;\Leftrightarrow\; \underbrace{p(z_1)v(z_1) + \cdots + p(z_n)v(z_n)}_{U(p)} \geq \underbrace{q(z_1)v(z_1) + \cdots + q(z_n)v(z_n)}_{U(q)}$$

- Recall our two earlier questions:
  1. What kind of preferences does Expected Utility represent?
  2. Can we identify the Bernoulli utility?

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- Recall our two earlier questions:
  1. What kind of preferences does Expected Utility represent? ✓
  2. Can we identify the Bernoulli utility?

Theorem (von Neumann & Morgenstern, 1944 (Part II))

The Bernoulli utility function $v : Z \to \mathbb{R}$ is unique in the following sense: if $\tilde{v} : Z \to \mathbb{R}$ is another Bernoulli utility function, then

there are $a \in \mathbb{R}$ and $b > 0$ such that $\tilde{v}(z) = a + bv(z)$ for all $z \in Z$

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  1. What kind of preferences does Expected Utility represent?
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Example

Take $Z = \{0, 4, 10\}$ and $\succeq$ that satisfy the vNM axioms.

How do we find the Bernoulli utility function?

1. Arbitrarily set highest $v(10) = 100$ and lowest utility $v(0) = 0$.
2. By monotonicity there is unique $\alpha \in (0, 1)$ such that
$$v(4) = \alpha v(10) + (1 - \alpha)v(0) = 100\alpha$$
3. Elicit this α, e.g., using BDM mechanism.

The Bernoulli utility is unique in the sense that intermediate utilities are uniquely determined, once we fix the maximum and minimum utilities.

- Positive linear transformations leave α unchanged (Part II of Theorem): $\tilde{v}(10) = 200$, $\tilde{v}(0) = 0$ will yield $\tilde{v}(4) = 200\alpha$.
- This is the relevant part for risk attitudes.
- What does this means?

- Choose between:
    - $p_1 = (0.25 \times 3000, 0.75 \times 0)$
    - $p_2 = (0.20 \times 4000, 0.80 \times 0)$
- Choose between:
    - $p_3 = (1 \times 3000)$
    - $p_4 = (0.80 \times 4000, 0.20 \times 0)$
- In experiments most subjects choose $\mathrm{p}_2 \succ \mathrm{p}_1$ and $\mathrm{p}_3 \succ \mathrm{p}_4$.
- This is not consistent with Expected Utility (Allais paradox)
- Why is it impossible to rationalize this data with an EU model?

- Choose between:
    - $p_1 = (0.25 \times 3000, 0.75 \times 0)$
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- In experiments most subjects choose $\mathrm{p}_2 \succ \mathrm{p}_1$ and $\mathrm{p}_3 \succ \mathrm{p}_4$.
- **First explanation:** IIA is violated
    - $p_1 = (0.25 \times p_3, 0.75 \times 0)$
    - $p_2 = (0.25 \times p_4, 0.75 \times 0)$

  Hence, we should have $p_2 \succ p_1 \Leftrightarrow p_4 \succ p_3$
- **Second explanation:** No EU model solves the system
    - Bernoulli utilities $v(4000) = 1$, $v(0) = 0$, $v(3000) = x$

$$\begin{aligned} U(p_2) = 0.20 &> 0.25x = U(p_1) \qquad (1) \\ U(p_3) = x &> 0.80 = U(p_4) \qquad (2) \end{aligned}$$

-
    - Multiply both sides of (1) by 4, and obtain $0.80 > x$
    - This contradicts (2)
- Are the two explanations equivalent?

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- How do you understand risk aversion in general?
- Choice between $p = (0.50 \times 10, 0.50 \times 0)$ and $q = (1 \times 5)$
- **Risk averse** people choose q:

$$\text{If } v(10) = 10 \text{ and } v(0) = 0, \text{ then } v(5) > 5$$

- **Risk neutral** people are indifferent

$$\text{If } v(10) = 10 \text{ and } v(0) = 0, \text{ then } v(5) = 5$$

- **Risk seeking** people choose p

$$\text{If } v(10) = 10 \text{ and } v(0) = 0, \text{ then } v(5) < 5$$

- Is this enough to classify people?

$$(0.50\times 10, 0.50\times 0) \succ (1\times 5) \overbrace{\Longleftrightarrow}^{?} (0.50\times 200, 0.50\times 100) \succ (1\times 150)$$

- We need the entire Bernoulli utility function
    1. Concave Bernoulli utility function means risk averse
    2. Linear Bernoulli utility function means risk neutral
    3. Convex Bernoulli utility function means risk seeking
- Risk attitudes may change for different wealth levels.

- Convexity does not change when we apply linear transformations. Why does this matter?

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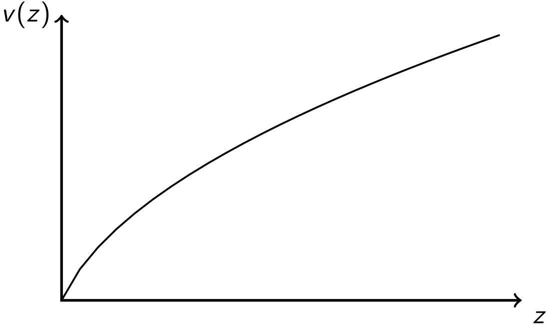

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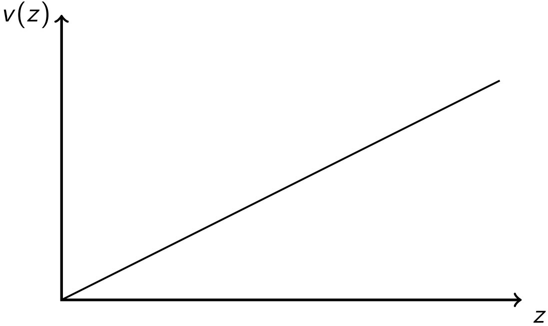

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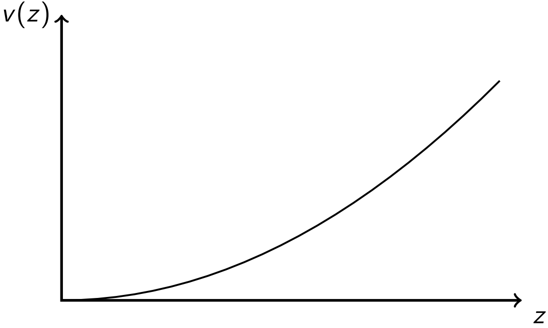

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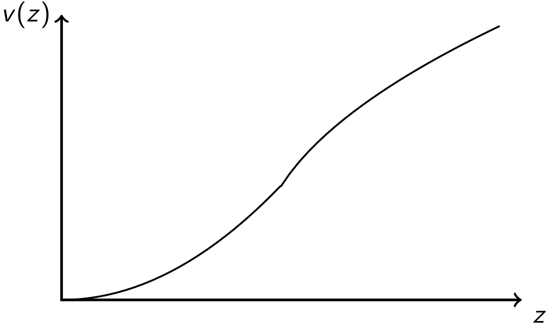

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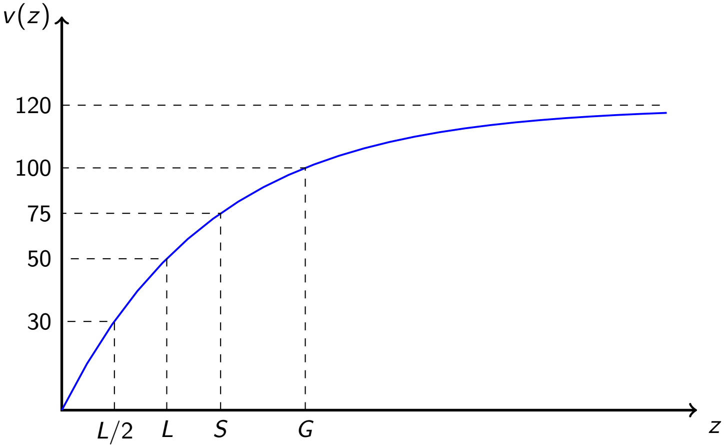
Theorem (Rabin, 2000)

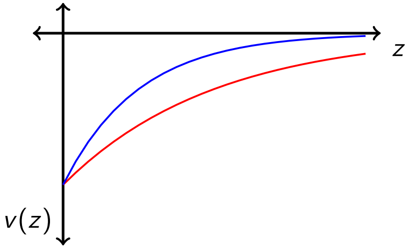
Moderate levels of risk aversion for small stakes implies extreme levels of risk aversion for large stakes.



- Sometimes, we assume specific (parametric) functional form for Bernoulli utility
- One common such family of utility functions is those that satisfy **Constant Absolute Risk Aversion (CARA)**:

$$v(z) = -e^{-\alpha z}$$

  where $\alpha > 0$ is a risk aversion coefficient.
- The red line is $\alpha = 1$ and the blue line is $\alpha = 2$



- We can then fit data into this functional form
  - What does this mean?
  - Why do we want to do this?

- At home you need to read Chapter 3 of the textbook.
- The sheet with the tasks will follow on canvas as usual.
- Same instructions as last week apply